

# $E_{7(7)}$ symmetry and dual gauge algebra of M–theory on a twisted seven–torus

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## ABSTRACT

We consider M–theory compactified on a twisted 7–torus with fluxes when all the seven antisymmetric tensor fields in four dimensions have been dualized into scalars and thus the  $E_{7(7)}$  symmetry is recovered. We find that the Scherk–Schwarz and flux gaugings define a “dual” gauge algebra, subalgebra of  $E_{7(7)}$ , where some of the generators are associated with vector fields which are dual to part of the original vector fields (deriving from the 3–form). In particular they are dual to those vector fields which have been “eaten” by the antisymmetric tensors in the original theory by the (anti–)Higgs mechanism. The dual gauge algebra coincides with the original gauge structure when the quotient with respect to these dual (broken) gauge generators is taken. The particular example of the S–S twist corresponding to a “flat group” is considered.

# 1 M–theory on twisted tori and its gauge structure

Compactification of superstring or M–theory on twisted tori [1, 2, 3, 4, 5, 6] in the presence of form–fluxes [7, 8] offers interesting models where a scalar potential for the moduli fields is obtained and gauge and supersymmetry breaking remove most (if not all) the flat directions of the original geometry.

From the point of view of the low–energy effective theory in lower dimensions (four dimensions in our case) the change in the geometry of the internal space results in a “massive” deformation of a certain supergravity theory, where the scalar potential and the Yukawa couplings are induced by the underlying gauge algebra structure of the gauged supergravity. The low–energy dynamics of M–theory compactified on a 7–torus is described by an effective  $D = 4$ ,  $N = 8$  (maximal) supergravity. We shall refer in the following to the maximal theory in four dimensions with 70 scalar fields (and, in the absence of twists or fluxes, with a manifest  $E_{7(7)}$  global invariance of the combined equations of motion and Bianchi identities [9]) as *standard*  $N = 8$ ,  $D = 4$  supergravity<sup>1</sup>. In the case of M–theory an interesting phenomenon occurs due to the fact that the ordinary compactification on a 7–torus with fluxes results in an unconventional maximal supergravity theory in  $D = 4$  where 7 of the 70 scalar fields have been replaced by antisymmetric tensors. The  $SL(7, \mathbb{R})$  assignment of the vectors and tensors from the M–theory compactification is:  $\mathbf{21} + \overline{\mathbf{7}}$  for the 28 vector fields and  $\mathbf{7}$  for the tensor fields. Denoting by  $A_{IJ\mu}$ ,  $A_\mu^I$  (in our notations the capital Latin indices label the internal directions of the 7-torus:  $I, J \dots = 1, \dots, 7$ ) the 1–form and by  $B_{I\mu\nu}$  the 2–form fields, their combined algebraic structure results in a “free differential algebra” (FDA) [12] which was studied in [5]. In particular the Scherk–Schwarz (S-S) structure constants  $\tau_{IJ}^K$  play the role, in this algebra, of a “magnetic” mass for the B–field. To show this let us consider the general 2–form structure which appear in the low–energy theory:

$$F^\Lambda = dA^\Lambda + \frac{1}{2} f_{\Sigma\Gamma}^\Lambda A^\Sigma A^\Gamma + m^{\Lambda I} B_I, \quad (1.1)$$

in which we have generically denoted by an upper index  $\Lambda$  the lower antisymmetric couple  $[IJ]$ :

$$F^\Lambda = F_{IJ}, \quad (1.2)$$

and we have identified the  $21 \times 7$  matrix  $m^{\Lambda I}$  with  $\tau_{KL}^I$ . If we assume this matrix to have rank  $r \leq 7$  then  $r$  of the (redefined) 2–forms  $B_I$ , denoted by  $\hat{B}_\alpha$ , appear in the following combination:

$$F^\alpha = dA^\alpha + m^{\alpha\beta} \hat{B}_\beta. \quad (1.3)$$

In the vacuum  $F^\alpha = 0$  and therefore  $d\hat{B}_\alpha = 0$ . Equation (1.3) indicates that the gauge fields  $A^\alpha$  make the  $\hat{B}_\alpha$  massive and that the quotient algebra of the  $A^\Lambda$  (with the  $A^\alpha$  modded out)

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<sup>1</sup>Note that this definition of  $N = 8$ ,  $D = 4$  standard supergravity comprises ungauged theories which, in spite of the common  $E_{7(7)}$  global symmetry at the level of field equations and Bianchi identities, have inequivalent Lagrangians with different global symmetry groups. These models therefore offer different choices of gauge symmetry which can be introduced [10, 11].

is an ordinary Lie algebra. The connection  $\Omega$  and structure (non vanishing commutators) of this algebra have the following form [3]:

$$\begin{aligned}\Omega &= A_\mu^I Z_I + A_{IJ\mu} W^{IJ}, \\ [Z_I, Z_J] &= \tau_{IJ}^K Z_K + g_{IJKL} W^{KL} ; \quad [Z_I, W^{JK}] = 2\tau_{IL}^{[J} W^{K]L},\end{aligned}\tag{1.4}$$

(here  $W^{IJ}$  are the remaining  $21 - r$  generators) with

$$\tau_{[IJ}^P \tau_{K]P}^M = \tau_{[IJ}^P g_{KLR]P} = 0.\tag{1.5}$$

The constraints (1.5) represent the Jacobi identities of the structure constants  $f_{\Sigma\Gamma}^\Lambda$  in eq.(1.1) as they come from the free differential algebra [5] closure condition or, equivalently, from the 4-form Bianchi identity in M-theory [3]. However condition (1.5) does not guarantee closure of the algebra (1.4). The latter indeed turns out to close under the general conditions (1.5) only if regarded either as part of the FDA (1.1) or, as we shall see, as part of a larger gauge algebra in the standard formulation of the theory.

In the particular case in which the only non vanishing entries of the  $\tau_{IJ}^K$  and  $g_{IJKL}$  tensors are  $\tau_{1i}^j = T_i^j$  (to be considered as an invertible  $6 \times 6$  matrix) and  $g_{1ijk}$  ( $i, j = 2, \dots, 7$ ), the quotient algebra is a 22-dimensional Lie algebra spanned by  $Z_1, Z_i, W_{ij}$  with structure constants  $T_i^j, g_{1ijk}$ . In the framework of standard maximal supergravity, the gauged theory corresponding to this choice of twist  $\tau$  and flux  $g$  was shown in [4] to coincide with the model originating from a  $D = 5 \rightarrow D = 4$  generalized S-S dimensional reduction [10, 13, 14], for a suitable choice of the S-S twist generator in  $E_{6(6)}$ . If moreover  $T_i^j = -T^j_i$  the gauge algebra defines a *flat group* in the language of [1] and the corresponding theory admits a Minkowski vacuum.

Note that the gauge algebra (1.4) does not arise from a gauging of standard  $N = 8, D = 4$  supergravity since, as we shall comment on later, it is not contained in the global symmetry algebra of the theory. The unusual situation is due to the fact that in the presence of antisymmetric tensor fields the original  $E_{7(7)}$  global symmetry of  $D = 4, N = 8$  supergravity is lost. In particular the  $E_{7(7)}$  isometries corresponding to the 7 scalars  $\tilde{B}^I$  which have been dualized into antisymmetric tensor fields have been replaced by tensor gauge symmetries with the implication that the generators  $W^{IJ}$  corresponding to the gauge fields  $A_{IJ\mu}$  have become abelian, as in the algebra (1.4).

The present note is organized as follows: In section 2 we consider the standard formulation of the four dimensional theory where the antisymmetric tensor fields  $B_{I\mu\nu}$  have been replaced by the scalars  $\tilde{B}^I$  and thus the  $E_{7(7)}$  global symmetry of the equations of motion and Bianchi identities is manifest. In this framework, we shall apply the general group theoretical analysis of gauged maximal supergravities developed in [11], to the construction of the gauge algebra arising from the presence of twist-tensor  $\tau_{IJ}^K$  and flux  $g_{IJKL}$ . This approach is based on the description of the most general gauging of the  $D = 4, N = 8$  theory in terms of an  $E_{7(7)}$ -covariant tensor  $\theta$  called *embedding matrix*, which defines the gauge generators as linear combinations of the global symmetry generators. The advantage of such description is that the  $E_{7(7)}$  invariance of the combined field equations and Bianchi identities of the gauged theory is restored provided  $\theta$  is transformed together with the fields of the model.

Supersymmetry requires  $\theta$  to transform in the **912** and closure of the gauge algebra inside  $E_{7(7)}$  implies further quadratic constraints in  $\theta$ . In this formalism the background quantities  $\tau_{IJ}^K$  and  $g_{IJKL}$  are identified with components of the embedding tensor. To be specific they correspond to the **140** (considering  $\tau$  to be traceless) and **35** in the branching of the **912** with respect to  $SL(7, \mathbb{R})$ . Group theory then determines the gauge algebra structure and we find that the second order constraints on  $\theta$  correspond precisely with the quadratic constraints (1.5) on  $\tau$  and  $g$ . As we shall see, the gauge generators consist not just of  $Z_I$  and  $W^{IJ}$ , but also of some “dual” generators  $W_{IJ}$  to be gauged by the magnetic vector fields  $\tilde{A}_\mu^{IJ}$  dual to  $A_{IJ\mu}$ . This seems in contradiction with the notion that a gauging which involves magnetic vector fields is inconsistent. Actually what the presence of “magnetic” generators  $W_{IJ}$  is telling us is that the electric fields to be described locally in the low-energy gauged theory are not  $A_\mu^I$ ,  $A_{IJ\mu}$ , but are defined after a symplectic rotation between  $A_{IJ\mu}$  and  $\tilde{A}_\mu^{IJ}$  (whose existence is guaranteed by the second order constraints [15]). This mechanism is described in detail in the case  $\tau \equiv \{\tau_{1i}^j\}$  and  $g \equiv \{g_{1ijk}\}$ , in which we show that this symplectic rotation yields electric vector fields transforming in the representation  $\overline{27} + \mathbf{1}$  of  $E_{6(6)}$ , thus confirming that the corresponding model can be alternatively obtained through a suitable  $D = 5 \rightarrow D = 4$  generalized S-S reduction.

In section 3 we review the FDA analysis of [5]. Finally we shall end with some concluding remarks.

## 2 Gaugings from M-theory with $E_{7(7)}$ symmetry

We devote the present section to the derivation, using the embedding tensor approach, of the gauge algebra structure in the  $D = 4, N = 8$  theory which originates from the presence of a non trivial twist tensor  $\tau_{MN}^P$  and 4-form flux  $g_{MNPQ}$ . We shall also consider the effect in terms of local symmetries of a flux over the volume of  $T^7$  of the 7-form  $\tilde{g}$  dual to  $g$ . The main result is the algebra (2.15). This analysis parallels the one of [5] in which the antisymmetric tensors are left undualized and the gauge structure originating from the compactification is encoded in a free differential algebra, to be reviewed in section 3.

In standard four dimensional maximal supergravity the electric and magnetic charges transform in the **56** of  $E_{7(7)}$  and, as anticipated in the introduction, the most general gauging can be described in terms of an embedding tensor [11]  $\theta_u^\sigma$  ( $u = 1, \dots, 56$  and  $\sigma = 1, \dots, 133$ ), which expresses the generators  $X_u$  of the gauge algebra  $\mathfrak{g}$  in terms of  $E_{7(7)}$  generators  $t_\sigma$ :

$$X_u = \theta_u^\sigma t_\sigma. \quad (2.1)$$

In this notation, since the index  $u$  runs over both electric and magnetic charges, consistency of the gauging requires the rank of  $\theta$  to be not greater than 28. Supersymmetry and closure of the gauge algebra inside  $E_{7(7)}$  require a linear and a quadratic condition in  $\theta$  respectively [11, 15]:

$$\theta \in \mathbf{912} \subset \mathbf{56} \times \mathbf{133}, \quad (2.2)$$

$$\theta_u^\sigma \theta_v^\gamma \mathbb{C}^{uv} = 0, \quad (2.3)$$

where  $\mathbb{C}^{uv}$  is the  $\text{Sp}(56, \mathbb{R})$ -invariant matrix. The last condition ensures that *there always exists a symplectic rotation acting on the index  $u$  as a consequence of which all the vectors associated with the generators  $X_u$  are electric (or all magnetic)*.

Let us start considering the branchings of the relevant  $E_{7(7)}$  representations with respect to the subgroup  $\text{SL}(7, \mathbb{R}) \times \text{O}(1, 1)_2$  (the subscript 2 will be used later to distinguish the corresponding grading from the charge with respect to a different  $\text{O}(1, 1)$  group):

$$\begin{aligned}
\mathbf{56} &\rightarrow \bar{\mathbf{7}}_{-3} + \mathbf{21}_{-1} + \bar{\mathbf{21}}_{+1} + \mathbf{7}_{+3}, \\
\mathbf{133} &\rightarrow \mathbf{7}_{-4} + \bar{\mathbf{7}}_{+4} + \bar{\mathbf{35}}_{-2} + \mathbf{35}_{+2} + \mathbf{48}_0 + \mathbf{1}_0, \\
\mathbf{912} &\rightarrow \mathbf{1}_{-7} + \mathbf{1}_{+7} + \mathbf{35}_{-5} + \bar{\mathbf{35}}_{+5} + (\bar{\mathbf{140}} + \bar{\mathbf{7}})_{-3} + (\mathbf{140} + \mathbf{7})_{+3} + \mathbf{21}_{-1} + \bar{\mathbf{21}}_{+1} + \\
&\quad \mathbf{28}_{-1} + \bar{\mathbf{28}}_{+1} + \mathbf{224}_{-1} + \bar{\mathbf{224}}_{+1}.
\end{aligned} \tag{2.4}$$

In the branching of the  $\mathbf{56}$  the  $\bar{\mathbf{7}}_{-3}$  and  $\mathbf{21}_{-1}$  define  $A_\mu^I$ ,  $A_{IJ\mu}$  respectively while  $\mathbf{7}_{+3}$  and  $\bar{\mathbf{21}}_{+1}$  their magnetic duals. In the branching of the adjoint representation of  $E_{7(7)}$  we denote by  $t_M^N$ ,  $t^{MNP}$ ,  $t_P$  the generators in the  $\mathbf{48}_0$ ,  $\mathbf{35}_{+2}$  and  $\bar{\mathbf{7}}_{+4}$  respectively (with an abuse of notation we characterize each generator by the representation of the corresponding parameter, this allows a simpler interpretation of the table below). The commutation relations between these generators is:

$$\begin{aligned}
[t_M^N, t^{PQR}] &= -3 \delta_M^{[P} t^{QR]N} ; [t_M^N, t_P] = \delta_P^N t_M, \\
[t_M^N, t_P^Q] &= \delta_P^N t_M^Q - \delta_M^Q t_P^N ; [t^{M_1 M_2 M_3}, t^{M_4 M_5 M_6}] = \epsilon^{M_1 \dots M_6 M} t_M.
\end{aligned} \tag{2.5}$$

Each representation in the branching of  $\mathbf{912}$  defines a different set of entries of  $\theta$  which can be switched on independently of the others and leads to a specific gauging. It is useful to arrange the above representations in a table as follows:

	$\mathbf{7}_{+3}$	$\bar{\mathbf{21}}_{+1}$	$\mathbf{21}_{-1}$	$\bar{\mathbf{7}}_{-3}$
$\bar{\mathbf{7}}_{+4}$	$\mathbf{1}$	$\bar{\mathbf{35}}$	$\mathbf{140} + \mathbf{7}$	$\bar{\mathbf{28}} + \bar{\mathbf{21}}$
$\mathbf{35}_{+2}$	$\bar{\mathbf{35}}$	$\mathbf{140}$	$\bar{\mathbf{21}} + \bar{\mathbf{224}}$	$\mathbf{21} + \mathbf{224}$
$\mathbf{48}_0$	$\mathbf{140} + \mathbf{7}$	$\bar{\mathbf{21}} + \bar{\mathbf{28}} + \bar{\mathbf{224}}$	$\mathbf{21} + \mathbf{28} + \mathbf{224}$	$\bar{\mathbf{140}} + \bar{\mathbf{7}}$
$\mathbf{1}_0$	$\mathbf{7}$	$\bar{\mathbf{21}}$	$\mathbf{21}$	$\bar{\mathbf{7}}$
$\bar{\mathbf{35}}_{-2}$	$\bar{\mathbf{21}} + \bar{\mathbf{224}}$	$\mathbf{21} + \mathbf{224}$	$\bar{\mathbf{140}}$	$\mathbf{35}$
$\mathbf{7}_{-4}$	$\mathbf{28} + \mathbf{21}$	$\bar{\mathbf{140}} + \bar{\mathbf{7}}$	$\mathbf{35}$	$\mathbf{1}$

The first row and column contain the representations in the branchings of  $\mathbf{56}$  and the  $\mathbf{133}$  respectively, while the bulk contains representations in the branching of  $\mathbf{912}$ . The table specifies the origin of the latter representations in the branching of the product  $\mathbf{56} \times \mathbf{133}$  and it should be read as “first row times first column gives bulk”. The grading of each entry of the table has been suppressed for the sake of simplicity, since it coincides with the sum of the gradings of the corresponding elements in the first row and column.<sup>2</sup>

<sup>2</sup>In principle there could have been a representation  $\mathbf{7}_{+3}$  in the slot  $\mathbf{35}_{+2} \times \bar{\mathbf{21}}_{+1}$  and a  $\bar{\mathbf{7}}_{-3}$  in the slot  $\bar{\mathbf{35}}_{-2} \times \mathbf{21}_{-1}$ . However the presence of these representations would be inconsistent with the corresponding table of branchings with respect to the  $\text{SL}(8, \mathbb{R})$  maximal subgroup of  $E_{7(7)}$ , which contains  $\text{GL}(7, \mathbb{R})$  (see table (6.2) of [11]).

To see what information can be gained from this table let us choose to restrict ourselves to the components of  $\theta$  in the representations  $\mathbf{140}_{+3}$ ,  $\overline{\mathbf{35}}_{+5}$  and  $\mathbf{1}_{+7}$  contained in the  $\mathbf{912}$ . The first two correspond (modulo multiplicative factors) to the tensors  $\tau_{MN}{}^P$  and  $g_{MNPQ}$  respectively, while the third is related to the flux of the dual 7-form  $\tilde{g}$  over  $T^7$ :

$$\begin{aligned}\mathbf{140}_{+3} &\leftrightarrow \tau_{MN}{}^P, \\ \overline{\mathbf{35}}_{+5} &\leftrightarrow g_{MNPQ}, \\ \mathbf{1}_{+7} &\leftrightarrow \tilde{g}_{M_1\dots M_7} = \tilde{g} \epsilon_{M_1\dots M_7}.\end{aligned}\tag{2.6}$$

We have a component of the embedding tensor, depending only on  $\tau$ , which intertwines between the electric charge in  $\mathbf{21}_{-1}$  and the  $E_{7(7)}$  generator  $t_M$  in the  $\overline{\mathbf{7}}_{+4}$ . This defines the following first set of gauge generators:

$$W_{MN} = \theta_{MN,}{}^P t_P = c_1 \tau_{MN}{}^P t_P.\tag{2.7}$$

Note that there are at most 7 independent  $W_{IJ}$  depending on the rank  $r \leq 7$  of the  $21 \times 7$  matrix  $\tau_{IJ}{}^K$ . Then we have two components  $\theta^{MN, PQR}$ ,  $\theta^{MN, P}$  of the embedding tensor with the electric index  $u$  in the same representation  $\overline{\mathbf{21}}_{+1}$ . They contract with the  $E_{7(7)}$  generators  $t^{PQR}$  in the  $\mathbf{35}_{+2}$  through the tensor  $\tau$  and with the generators  $t_P$  in the  $\overline{\mathbf{7}}_{+4}$  through  $g$ . These components define the following generators:

$$W^{MN} = \theta^{MN, PQR} t^{PQR} + \theta^{MN, P} t_P = b_1 \tau_{PQ}^{[M} t^{N]PQ} + b_2 \epsilon^{MNM_1\dots M_4P} g_{M_1\dots M_4} t_P.\tag{2.8}$$

Finally there are three more components of  $\theta$  which intertwine between the representations  $\mathbf{7}_{+3} \in \mathbf{56}$  and  $\mathbf{48}_0 \in \mathbf{133}$  through the tensor  $\tau$ , between the representations  $\mathbf{7}_{+3} \in \mathbf{56}$  and  $\mathbf{35}_{+2} \in \mathbf{133}$  through the tensor  $g$  and between the  $\mathbf{7}_{+3} \in \mathbf{56}$  and the  $\overline{\mathbf{7}}_{+4} \in \mathbf{133}$  through the tensor  $\tilde{g}$ . They define the last set of gauge generators:

$$\begin{aligned}Z_M &= \theta_{M, M_1 M_2 M_3} t^{M_1 M_2 M_3} + \theta_{M, N}{}^P t_P{}^N + \theta_{M,}{}^N t_N = a_1 g_{M M_1 M_2 M_3} t^{M_1 M_2 M_3} + \\ &\quad a_2 \tau_{MN}{}^P t_P{}^N + a_3 \tilde{g} t_M.\end{aligned}\tag{2.9}$$

The constraints (2.3) imply the following conditions:

$$\begin{aligned}\theta_{MN,}{}^P \theta^{MN, RST} &= 0 \Rightarrow \tau_{[MN}{}^P \tau_{Q]P}{}^R = 0 \\ \theta_{MN,}{}^P \theta^{MN, Q} - \theta_{MN,}{}^Q \theta^{MN, P} &= 0 \Rightarrow \tau_{[MN}{}^P g_{M_1 M_2 M_3]P} = 0.\end{aligned}\tag{2.10}$$

Taking into account equations (2.7) and (2.8), we see that the generators  $W^{MN}$  and  $W_{MN}$  are not linearly independent since they satisfy the two constraints:

$$\begin{aligned}\tau_{[PQ}{}^N W_{R]N} &= 0 \\ b_2 \epsilon^{M_1 M_2 M_3 M_4 PQR} g_{M_1 M_2 M_3 M_4} W_{QR} &= c_1 \tau_{ST}{}^P W^{ST}.\end{aligned}\tag{2.11}$$

In particular this means that if  $r$  is the rank of the  $21 \times 7$  matrix  $\tau_{MN}{}^P$  only  $r$   $W_{PQ}$  generators and  $(21 - r)$   $W^{PQ}$  generators are linearly independent.

The previous analysis indicates that the gauge connection has the following form:

$$\Omega_{g\mu} = A_\mu^M Z_M + A_{MN\mu} W^{MN} + \tilde{A}_\mu^{MN} W_{MN}.\tag{2.12}$$

where the restrictions (2.11) are understood. Although this gauging involves the vectors  $A_{MN\mu}$  together with their duals  $\tilde{A}_\mu^{MN}$ , the conditions (2.11) and (2.10) guarantee that no more than 28 independent linear combinations of them can take part to the minimal couplings, namely that there exists a symplectic frame in which all the  $A_{MN\mu}$  and  $\tilde{A}_\mu^{MN}$  involved in this gauging are electric. An other way of stating this is that *the gauging chooses its own symplectic frame*. This symplectic frame is in general different from the  $\text{GL}(7, \mathbb{R})$  one in which the magnetic charges (vector fields) transform in the  $\bar{\mathbf{7}}_{-3} + \mathbf{21}_{-1}$ . For instance in the case to be discussed in the next subsection the natural frame is the  $E_{6(6)} \times O(1, 1)$  and the corresponding gauging coincides with the one originating from  $D = 5 \rightarrow D = 4$  S-S reduction [13], for a suitable choice of parameters.

The general structure of the algebra is:

$$\begin{aligned}
[Z_M, Z_N] &= \alpha \tau_{MN}^P Z_P + \beta g_{MNPQ} W^{PQ} + \rho \tilde{g} W_{MN}, \\
[Z_M, W^{PQ}] &= \gamma \tau_{MR}^{[P} W^{Q]R} + \sigma g_{MM_1 M_2 M_3} \epsilon^{M_1 M_2 M_3 PQRS} W_{RS}, \\
[Z_M, W_{PQ}] &= \delta \tau_{PQ}^L W_{ML} \\
[W^{IJ}, W^{KL}] &= -\frac{\lambda}{2} \tau_{I_1 I_2}^{[K} W_{I_3 I_4} \epsilon^{L] I J I_1 \dots I_4}, \\
[W^{IJ}, W_{KL}] &= [W_{IJ}, W_{KL}] = 0.
\end{aligned} \tag{2.13}$$

Closure in  $E_{7(7)}$  implies the following relations between the coefficients:

$$\begin{aligned}
a_1 &= 8\alpha; \quad a_2 = \alpha = \frac{\gamma}{2}; \quad b_1 = 24 \frac{\alpha^2}{\beta} = \frac{b_2}{2}, \\
c_1 &= -96 \frac{\alpha^3}{\beta\sigma}; \quad \frac{\lambda}{\sigma} = \frac{6\alpha}{\beta}; \quad \delta = \alpha; \quad \sigma = \alpha.
\end{aligned} \tag{2.14}$$

In particular we can choose  $\alpha = \beta = \rho = \sigma = 1$  and  $a_3 = c_1/a_2$  and eqs. (2.13) will read:

$$\begin{aligned}
[Z_M, Z_N] &= \tau_{MN}^P Z_P + g_{MNPQ} W^{PQ} + \tilde{g} W_{MN}, \\
[Z_M, W^{PQ}] &= 2 \tau_{MR}^{[P} W^{Q]R} + g_{MM_1 M_2 M_3} \epsilon^{M_1 M_2 M_3 PQRS} W_{RS}, \\
[Z_M, W_{PQ}] &= \tau_{PQ}^L W_{ML} \\
[W^{IJ}, W^{KL}] &= -3 \tau_{I_1 I_2}^{[K} W_{I_3 I_4} \epsilon^{L] I J I_1 \dots I_4}, \\
[W^{IJ}, W_{KL}] &= [W_{IJ}, W_{KL}] = 0.
\end{aligned} \tag{2.15}$$

The non-vanishing commutator between two  $W^{IJ}$  generators follows from the embedding of the gauge algebra inside  $E_{7(7)}$ . In particular it is a consequence of the last commutator in (2.5), where  $t_M$  are the isometry generators associated with the 7 axions dual to the antisymmetric tensor fields  $B_I$ . If on the other hand these tensors were left undualized, the scalar manifold would not have had the isometries  $t_M$ . As a consequence of this, the last commutator in (2.5) would vanish and the generators  $W^{IJ}$  would be abelian. This is an example of the phenomenon called *dualization of dualities* discussed in [16].

Note that the generators  $W_{IJ}$  define an abelian ideal  $\mathfrak{I}$  inside the gauge algebra  $\mathfrak{g}$ . If we consider the quotient algebra

$$\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{I}, \tag{2.16}$$

it has the structure described in (1.4) and, as we shall see, is also realized in a subsector of the FDA associated with the theory with undualized antisymmetric tensor fields [5]. Such subsector consists of the massless forms which survive in the effective theory after the Higgs mechanism between 1- and 2-forms has taken place [13].

## 2.1 Symplectic frame and the S-S gauging

As previously pointed out, the second order constraints (2.10) guarantee that no more than 28 independent combinations out of  $A_\mu^M$ ,  $A_{MN\mu}$ ,  $\tilde{A}_\mu^{MN}$  are involved in the gauging and thus define the actual elementary vector fields of the model. These combinations are defined by the twist-tensor  $\tau$ . Let us denote as usual by  $r$  the rank of  $\tau_{MN}^P$  as a  $21 \times 7$  matrix. The counting of the elementary vector fields proceeds as follows. Let us denote now by  $A_{P\mu}$   $r$  independent components of the  $A_{MN\mu}$  vectors defined as follows:

$$A_{MN\mu} = -2\tau_{MN}^P A_{P\mu} + \mathring{A}_{MN\mu}, \quad (2.17)$$

$\mathring{A}_{MN\mu}$  being the remaining  $21 - r$  components. One can check that the  $r$  vectors  $A_{P\mu}$  can always be reabsorbed in a redefinition of the vectors  $\tilde{A}_\mu^{MN}$ . Indeed, by using eqs. (2.11) and (2.17), the gauge connection (2.12) can be rewritten in the following form:

$$\Omega_{\mathfrak{g}\mu} = A_\mu^M Z_M + \mathring{A}_{MN\mu} W^{MN} + \tilde{A}_\mu'^{MN} W_{MN}, \quad (2.18)$$

where

$$\tilde{A}_\mu'^{MN} = \tilde{A}_\mu^{MN} + A_{P\mu} \epsilon^{MNM_1\dots M_4P} g_{M_1\dots M_4}. \quad (2.19)$$

From the expression of the  $W_{MN}$  generators in terms of  $t_P$  given in eq. (2.7), we see that only  $r$  independent combinations  $\tilde{A}_\mu^P = \tau_{MN}^P \tilde{A}_\mu'^{MN}$  of the 21  $\tilde{A}_\mu'^{MN}$ , take part in the minimal couplings. Thus the vector fields actually involved in the gauging sum up to 28 and consist in the 7 Kaluza-Klein vectors  $A_\mu^M$ , the  $21 - r$  vectors  $\mathring{A}_{MN\mu}$  and the  $r$  vectors  $\tilde{A}_\mu^P$ . These latter have Stueckelberg-like couplings to  $r$  of the scalars  $\tilde{B}^M$ , as a consequence of which they acquire mass through the Higgs mechanism.

We may dualize the scalars  $\tilde{B}^M$  back to the tensors  $B_{M\mu\nu}$ . In this case consistency of the theory requires a corresponding dualization of the vector fields, associated with a electric-magnetic duality rotation. The  $r$  vectors  $\tilde{A}_\mu^P$  are dualized into the  $A_{P\mu}$  components of the  $A_{MN\mu}$  vectors, defined in eq. (2.17). These latter enter the Lagrangian in the following combination with the antisymmetric tensor fields:

$$dA_{MN} + \tau_{MN}^P B_P = \tau_{MN}^P (dA_P + B_P) + d\mathring{A}_{MN}, \quad (2.20)$$

In the dual theory therefore the  $r$  vectors  $A_P$  give mass to  $r$  of the tensors  $B_P$  by means of an anti-Higgs mechanism.

As an example let us consider now the gauging induced by the following choice on non vanishing components of  $\tau$ ,  $g$ :

$$\tau_{1m}^n ; g_{1mnp} \quad (m, n, p = 2, \dots, 7). \quad (2.21)$$

These components are defined by the branching of the representations  $\mathbf{140}_{+3}$  and  $\overline{\mathbf{35}}_{+5}$  with respect to the  $\text{SL}(6, \mathbb{R}) \times \text{O}(1, 1)_1$  subgroup of  $\text{SL}(7, \mathbb{R})$ . However the same components are also defined by the branching with respect to  $\text{GL}(6, \mathbb{R}) \subset E_{6(6)}$  of the  $E_{6(6)} \times \text{O}(1, 1)_3$ -representation  $\mathbf{78}_{+3}$  ( $E_{6(6)} \times \text{O}(1, 1)_3$  being a subgroup of  $E_{7(7)}$ ), contained inside  $\mathbf{912}$  and defining the embedding tensor for the generalized S-S gauging [11] (the graviphoton originating from the  $D = 5 \rightarrow D = 4$  reduction is given a  $\text{O}(1, 1)_3$ -grading  $-3$ ). To show this let us start branching the  $\mathbf{140}_{+3}$  and  $\overline{\mathbf{35}}_{+5}$  with respect to  $\text{SL}(6, \mathbb{R}) \times \text{O}(1, 1)_1 \times \text{O}(1, 1)_2$ :

$$\begin{aligned} \mathbf{140}_{+3} &\rightarrow \overline{\mathbf{6}}_{(-1, +3)} + \mathbf{15}_{(-8, +3)} + \mathbf{35}_{(+6, +3)} + \mathbf{84}_{(-1, +3)} \\ \overline{\mathbf{35}}_{+5} &\rightarrow \mathbf{20}_{(+3, +5)} + \overline{\mathbf{15}}_{(-4, +5)} \end{aligned} \quad (2.22)$$

The components in (2.21) correspond to the representations  $\mathbf{35}_{(+6, +3)}$  and  $\mathbf{20}_{(+3, +5)}$  respectively. The representations  $\mathbf{35}$  and  $\mathbf{20}$  also appear in the decomposition of the  $\mathbf{78}$  of  $E_{6(6)}$  with respect to its subgroup  $\text{SL}(6, \mathbb{R})$ . To prove that the  $\mathbf{35}_{(+6, +3)}$  and  $\mathbf{20}_{(+3, +5)}$  belong to the decomposition of the S-S embedding tensor defined by the  $\mathbf{78}_{+3}$  we show that they have the right grading with respect to  $\text{O}(1, 1)_3$ , namely  $+3$ . The relation between the  $\text{O}(1, 1)_3$ -grading  $k_3$  and the  $\text{O}(1, 1)_{1,2}$ -gradings  $k_1, k_2$  can be found to be:

$$k_3 = \frac{1}{7} (2k_1 + 3k_2). \quad (2.23)$$

Applying this formula to the two representations we find that their  $\text{O}(1, 1)_3$ -grading is indeed  $+3$ .

The gauging induced by the components (2.21) coincides with a S-S gauging obtained by reducing the maximal theory in  $D = 5$  with a twist matrix given by:

$$Z_1 = \tau_{1m}{}^n t_n{}^m + g_{1mnp} t^{mnp} \in E_{6(6)}, \quad (2.24)$$

where  $t_n{}^m$  are the generators of the  $\text{SL}(6, \mathbb{R})$  subgroup of  $E_{6(6)}$  and  $t^{mnp}$  are generators in the  $\mathbf{20}_{+1}$  in the decomposition of the adjoint of  $E_{6(6)}$  with respect to  $\text{GL}(6, \mathbb{R})$ :

$$\mathbf{78} \rightarrow \mathbf{35}_0 + \mathbf{1}_0 + \mathbf{20}_{+1} + \mathbf{20}_{-1} + \mathbf{1}_{+2} + \mathbf{1}_{-2}. \quad (2.25)$$

Let us comment now on the symplectic frame corresponding to this gauging, namely the frame in which all the gauge generators are electric. If we arrange the generators  $X_u$  in a symplectic vector we have:

$$X_u = \begin{pmatrix} Z^1 = 0 \\ Z^m = 0 \\ W_{mn} = 0 \\ W_{1m} \neq 0 \\ Z_1 \neq 0 \\ Z_m \neq 0 \\ W^{mn} \neq 0 \\ W^{1m} = 0 \end{pmatrix}. \quad (2.26)$$

The symplectic rotation needed to define the right symplectic frame is effected by switching the  $\mathbf{6}$  of  $W_{1m}$  with the  $\overline{\mathbf{6}}$  of the  $W^{1m}$ . Therefore the new electric charges transform in the  $\mathbf{1} + \overline{\mathbf{15}} + 2 \times \mathbf{6}$  which complete the  $\mathbf{27} + \mathbf{1}$  of  $E_{6(6)}$ . This shows that the gauged Lagrangian induced by the torsion/flux components (2.21) coincides with the generalized S-S Lagrangian also as far as the symplectic frame is concerned.

### 3 The FDA approach

In this section we give a short resumé of the results obtained in [5], to be compared with the results illustrated in the previous section. The FDA obtained from M-theory compactification on twisted tori with form-fluxes is given by:

$$\begin{aligned} dA^I + \frac{1}{2} \tau_{JK}^I A^J \wedge A^K &= 0 \\ dA_{IJ} + 2 \delta_{[I}^{[L} \tau_{J]K}^{M]} A^K \wedge A_{LM} + \frac{1}{2} g_{IJKL} A^K \wedge A^L + \tau_{IJ}^L B_L &= 0 \\ dB_I + \tau_{IL}^J A^L \wedge B_J + \frac{1}{6} g_{IJKL} A^J \wedge A^K \wedge A^L &= 0. \end{aligned} \quad (3.1)$$

where integrability requires:

$$\tau_{[MN}^P \tau_{Q]P}^R = 0 \quad \tau_{[MN}^P g_{M_1 M_2 M_3]P} = 0 \quad (3.2)$$

Let us denote by  $\mathcal{A}^{(k)}$  a FDA generated by  $p$ -forms of degree  $p \leq k$ . Then a general theorem on the FDAs [12] guarantees that this differential algebra has a unique decomposition as the semi-direct sum of two algebras :

$$\mathcal{A}^{(k)} = \mathcal{M}^{(k)} \oplus_s \mathcal{C}^{(k)} \quad (3.3)$$

where the “contractible algebra”  $\mathcal{C}^{(k)}$  has the structure

$$d\mathcal{C}^{(k)} \subset \mathcal{C}^{(k+1)} \quad (3.4)$$

while the “minimal algebra”  $\mathcal{M}^{(k)}$  has the structure:

$$d\mathcal{M}^{(k)} \subset \mathcal{M}^{(k)} \wedge \mathcal{M}^{(k)} \quad (3.5)$$

An example of this decomposition has been given in [5] in the case of the S-S gauging discussed earlier and defined by  $\tau \equiv \tau_{1i}^j = T_i^j$ . Here we further require  $T_i^j = -T^j_i$ . Defining

$$\begin{aligned} \hat{B}_i &= B_i - T^j_k T_i^{-1\ell} A_{\ell j} \wedge A^k + A_{1i} \wedge A^1 + \frac{1}{2} T_i^{-1j} g_{j k \ell 1} A^k \wedge A^\ell, \\ \hat{B}_1 &= B_1 + A_{1i} \wedge A^i, \end{aligned}$$

the FDA becomes:

$$\begin{aligned}
dA^1 &= 0 \\
dA^i + T^i_j A^1 \wedge A^j &= 0 \\
dA_{ij} + 2T^k_{[i} A^1 \wedge A_{j]k} + g_{1ijk} A^1 \wedge A^k &= 0 \\
d\hat{B}_1 + T^i_j A^k \wedge A^j \wedge A_{ki} - \frac{1}{3} g_{1ijk} A^i \wedge A^j \wedge A^k &= 0 \\
dA_{1i} + T^j_i \hat{B}_j &= 0 \Rightarrow d\hat{B}_j = 0, \quad (3.6)
\end{aligned}$$

where the first four equations define the minimal algebra  $\mathcal{M}^{(2)}$  of which the first three correspond to the Lie algebra, while the fifth equation defines the contractible algebra.

The physical interpretation of the contractible algebra  $\mathcal{C}^{(2)}$  is that it consists in those 2-forms and 1-forms which are involved in the Higgs mechanism: the tensors  $\hat{B}_j$  become massive by eating the vectors  $A_{1i}$ . As far as the Lie algebra contained in  $\mathcal{M}^{(2)}$  is concerned, it reproduces the structure (1.4) in this special example. This latter property is however general and allows us to make the following statement about the connections between the FDA approach and the gauged supergravity analysis of the previous section: The gauge Lie algebra contained in  $\mathcal{M}^{(2)}$  has the same structure (1.4) as the quotient algebra (2.16).

## 4 Conclusions

In the  $E_{7(7)}$  four-dimensional formulation of M-theory, in the  $SL(7, \mathbb{R})$ -basis, we have seen that the gauge algebra, when the S-S twist  $\tau_{IJ}^K$  is non-vanishing, contains both  $W^{KL}$  and  $W_{KL}$  generators. The total number of these generators is 21 while the number of  $W_{IJ}$  is bound to be less or equal to 7, depending on the rank of the  $21 \times 7$  matrix  $\tau_{IJ}^K$ . When  $\tau_{IJ}^K = 0$  and  $W_{KL} = 0$ , the dual algebra (2.15) and the original algebra (1.4) coincide. This is expected because in this case the antisymmetric tensors have no magnetic mass terms which couple them to the gauge generators. However when  $\tau_{IJ}^K \neq 0$ , depending on its rank  $r$  (as a  $21 \times 7$  matrix),  $21 - r$  generators are of type  $W^{KL}$  and  $r$  are of type  $W_{KL}$ , with a non-vanishing commutation given in (2.15) and depending on the  $\tau$  matrix. This is the dual algebra of the original M-theory compactification on a twisted torus with fluxes [3, 4, 5]. We also note that the gauge vectors  $\tilde{A}_\mu^{IJ}$ , corresponding to the  $W_{IJ}$  generators, are  $r$  in number and correspond to the vectors which are eaten by the antisymmetric tensors in the FDA formulation. This result is not surprising in view of the dynamics of supergravity coupled to antisymmetric tensor fields studied in [17]-[24].

Indeed when massive antisymmetric tensors are dualized into massive vectors, each of which can be written in terms of a massless vector plus a scalar field through the Stueckelberg mechanism, a symplectic rotation occurs for the vector fields which is equivalent to the appearance of the generators  $W_{IJ}$  in the dual gauge algebra structure. Note that this is also implied by the compatibility with the  $E_{7(7)}$  symmetry which is manifest in the dual formulation.

The two gauge algebra structures, namely (1.4) and the dual (2.15), just coincide if one

restricts oneself to the smaller gauge algebra resulting from the quotient by the *abelian ideal* generated by the additive  $r$  generators  $W_{IJ}$ .

It is an interesting problem, in the presence of gauge couplings, to carry out the dualization of the M-theory Lagrangian in order to show the equivalence of the two formulations.

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